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Lipschitz Conditions on Uniform Approximation Operators

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INTRODUCTION

Consider $C(X)$, the space of continuous, real-valued functions on a compact Hausdorff space X, with the uniform norm. It is assumed that $C(X)$ contains a finite dimensional subspace G with the Chebyshev property (i.e., no element of G other than $g = 0$ has n (distinct) zeros on X, where n is the dimension of G). Let f be a member of $C(X)$ which is not contained in G. Then we are guaranteed the existence of a unique element g^* of G such that

$$\|f - g^*\| \leqslant \|f - g\|$$

for every $g \in G$.

It is easy to show that if $\|f - g\| \approx \|f - g^*\|$, then $g \approx g^*$ (in the sense that a sequence $\{g_j\}$ satisfying $\|f - g_j\| \to \|f - g^*\|$, as $j \to \infty$, also satisfies $g_j \to g^*$).

The question arises, however, what is the nature of this convergence? A result of Newman and Shapiro [p. 680] is that there exists a constant K such that

$$\|g - g^*\| \leqslant K(\|f - g\| - \|f - g^*\|). \tag{1}$$

An immediate consequence (first proved independently by Freud [p. 162]) is that for f_1, another member of $C(X) \sim G$, and $g_1{}^*$, the best approximation to f_1 from G, the following inequality holds:

$$\|g^* - g_1{}^*\| \leqslant K_f \|f - f_1\|. \tag{2}$$

The constant K_f depends upon f, and, as Cheney [p. 82] has shown, K_f can be taken as $2K$ (K as in (1)).

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This paper explores various notions relating to inequalities (1) and (2). In Sec. 1, it will be shown how (1) can be used to reduce the domain of approximation problems and also that an appropriate K can be determined without requiring knowledge of g^* (but only of $\|f - g^*\|$). In Sec. 2, it will be shown that if X is a finite set, the constant in (2) can be made independent of f, but such is not the case if X is infinite. In Sec. 3, it will be shown that a suitable K can be found as the result of solving $n - 1$ interpolation problems. This is used in Sec. 4 to determine K in a specific case. In the concluding Sec. 5, we take $C(X)$ to be the space of continuous, complex valued functions on X, in which case, in general, (1) does not hold.

1. An Application of K and an Alternative Determination

It is clear that if some K satisfies (1), then every larger value of K also will. We may seek the minimal such K and label it K_0. Then it is clear that K_0 satisfies

$$K_0 = \sup_{G \sim \{g^*\}} \frac{\|g - g^*\|}{\|f - g\| - \|f - g^*\|} .$$

Letting the extremal set of the residual $f - g^*$ be labeled E (i.e.,

$$E = \{x \in X : |f(x) - g^*(x)| = \|f - g^*\|\},$$

then g^* is also the unique best uniform approximation from G to f on D, for any compact subset D of X such that $E \subset D$. The ability to reduce consideration to a subset of X may substantially reduce the scope of the problem. The following theorem shows that such sets D can be constructed when estimates of K_0 and $\|f - g^*\|$ are given.

Theorem 1. *Given $K \geq K_0$, $\tau \leq \|f - g^*\|$, and any $g \in G$, let*

$$m = \tau - K(\|f - g\| - \tau).$$

Then g^*, the best uniform approximation to f on X, is also the best uniform approximation to f on $D = \{x \in X : |f(x) - g(x)| \geq m\}$.

Proof. It suffices to show $E \subset D$. For $x \in E$, $|f(x) - g^*(x)| = \|f - g^*\|$ and

$$\begin{aligned} |f(x) - g(x)| &\geq |f(x) - g^*(x)| - |g^*(x) - g(x)| \\ &\geq \|f - g^*\| - \|g^* - g\| \\ &\geq \|f - g^*\| - K_0(\|f - g\| - \|f - g^*\|) \\ &\geq \tau - K(\|f - g\| - \tau) = m, \end{aligned}$$

Thus $x \in D$.

It is clear that the ability of such sets D to reduce the scope of the problem is related to how good the estimates K and τ are of K_0 and $\|f - g^*\|$, respectively, and also how close g is to g^*. For $g = g^*$ and $\tau = \|f - g^*\|$ the quantity $m = \|f - g^*\|$, hence the set D is exactly E. However, if $K(\|f - g\| - \tau) \geq \tau$, then $m \leq 0$, and D is X, hence no improvement is made.

The underestimate τ, for $\|f - g^*\|$ can be obtained from L_2 approximation theory. Given a measure μ on X such that G and f are contained in $L_2(X, \mu)$ then if $\bar{g}$ is a best approximation to f from G with respect to the norm induced by μ, it follows that

$$\int_X |f - \bar{g}|^2 \, d\mu \leq \int_X |f - g^*|^2 \, d\mu \leq \|f - g^*\|^2 \mu(X).$$

Hence

$$\tau = \left(\frac{\int_X |f - \bar{g}|^2 \, d\mu}{\mu(X)}\right)^{1/2}$$

provides a suitable underestimate for $\|f - g^*\|$.

The following characterization shows that K_0 can be determined with knowledge only of $\|f - g^*\|$.

THEOREM 2. *Let* $G^* = \{g \in G : \|f - g\| > \|f - g^*\|\}$. *Then*

$$K_0 = \sup_{G^*} \frac{\|g_1 - g_2\|}{\|f - g_1\| + \|f - g_2\| - 2\|f - g^*\|}.$$

Proof. It is clear that $G^* = G \sim \{g^*\}$ and from the definition

$$\begin{aligned} K_0 &= \sup_{g_1 \in G \sim \{g^*\}} \frac{\|g_1 - g^*\|}{\|f - g_1\| - \|f - g^*\|} \\ &= \sup_{g_1 \in G \sim \{g^*\}} \frac{\|g_1 - g^*\|}{\|f - g_1\| + \|f - g^*\| - 2\|f - g^*\|} \\ &\leq \sup_{g_1, g_2 \in G \sim \{g^*\}} \frac{\|g_1 - g_2\|}{\|f - g_1\| + \|f - g_2\| - 2\|f - g^*\|}. \end{aligned}$$

However, for $g_1, g_2 \in G \sim \{g^*\}$

$$\begin{aligned} 0 &\neq \|f - g_1\| + \|f - g_2\| - 2\|f - g^*\| \\ &= (\|f - g_1\| - \|f - g^*\|) + (\|f - g_2\| - \|f - g^*\|) \\ &\geq \frac{1}{K_0}(\|g_1 - g^*\| + \|g_2 - g^*\|) \\ &\geq \frac{1}{K_0}\|g_1 - g_2\|. \end{aligned}$$

Thus, for every pair $g_1, g_2 \in G \sim \{g^*\}$

$$K_0 \geq \frac{\| g_1 - g_2 \|}{\| f - g_1 \| + \| f - g_2 \| - 2 \| f - g^* \|}. \quad \blacksquare$$

The preceding yields underestimates of K_0 but Theorem 1 and general use require an overestimate. However, if a sequence of approximants $\{g^*\}$ is output by some algorithm, the ratio in Theorem 2 could be calculated using g^k and g^{k+1} and any overestimate for $\| f - g^* \|$ (e.g., $\| f - g^{k+1} \|$). From the nature of this sequence one might guess K_0 and perhaps apply some factor (depending upon one's cautiousness) to "insure" a $K \geq K_0$.

2. Lipschitz Condition on the Approximation Operator

Inequality (2) bounds the difference of two best approximations by a constant times the difference of the two respective functions being approximated. Unfortunately, the constant depends upon one of the functions. It is the purpose of this section to show that in the case of finite sets X, this constant can be made independent of the function and hence the best approximation operator is Lipschitz continuous. It will also be shown that such is not the case when X is not finite.

For this purpose we require several results found in Cheney [pp. 80–82] and summarized in the following lemma.

LEMMA 1. *Let* $G_1 = \{g \in G : \| g \| = 1\}$ *and*

$$K = (\min_{g \in G_1} \max_{x \in E} (\operatorname{sgn}[f(x) - g^*(x)]) \cdot g(x))^{-1}$$

Then $K > 0$,

$$\| g - g^* \| \leq K(\| f - g \| - \| f - g^* \|),$$

and

$$\| g^* - g_1^* \| \leq 2K \| f - f_1 \|$$

(*where* f_1 *and* g_1^* *are as in* (2)).

THEOREM 3. *Assume* X *is a finite point set, then there is a constant* K^* (*depending only upon* X *and* G) *such that for any* $f_1, f_2 \in C(X)$, *the best approximations* g_1^* *and* g_2^* *to* f_1 *and* f_2, *respectively, satisfy*

$$\| g_1^* - g_2^* \| \leq K^* \| f_1 - f_2 \|. \tag{3}$$

Proof. If $f_1, f_2 \in G$, the theorem holds with $K^* \geq 1$; and if $f_1 \in G$ but $f_2 \notin G$ with $K^* \geq 2$. Henceforth it is assumed that $f_1, f_2 \notin G$ and $K^* \geq 2$.

For fixed f_1, from Lemma 1, (3) is satisfied with

$$K = 2(\min_{g \in G_1} \max_{x \in E} (\operatorname{sgn}[f_1(x) - g_1^*(x)]) \cdot g(x))^{-1}.$$

This K is obviously a continuous function of f_1 and thus assumes a maximum on the compact unit ball of $C(X)$ (recall the finiteness of X implies $C(X)$ is finite dimensional). Label this maximum K^*, then for any f_1 such that $\|f_1\| = 1$

$$\|g_1^* - g_2^*\| \leq K^* \|f_1 - f_2\|.$$

In general, if f_1 and f_2 are both the zero function, then so are g_1^* and g_2^*, thus (3) holds independent of K^*. Otherwise, we may assume $\|f_1\| \neq 0$ and hence $f_1/\|f_1\|$ is of unit norm. The best approximation to this function is $g_1^*/\|f_1\|$ and similarly the best approximation to $f_2/\|f_1\|$ is $g_2^*/\|f_1\|$. Thus

$$\begin{aligned} \|g_1^* - g_2^*\| &= \|f_1\| \, \|g_1^*/\|f_1\| - g_2^*/\|f_1\|\| \\ &\leq K^* \|f_1\| \, \|f_1/\|f_1\| - f_2/\|f_1\|\| \\ &= K^* \|f_1 - f_2\|. \end{aligned}$$

We now move to the case where X is infinite. In the preceding case the finite nature of X was used only to guarantee that the unit sphere of $C(X)$ was compact. We shall see that except possibly in the case where $n = 1$, X having infinite points implies no inequality like (3) holds.

THEOREM 4. *If X is infinite and $n \geq 2$, then there exists no constant K^* such that for every f_1 and $f_2 \in C(X)$*

$$\|g_1^* - g_2^*\| \leq K^* \|f_1 - f_2\|.$$

Proof. It suffices to show that given any $\epsilon > 0$ there exist f_1 and f_2 such that $\|f_1 - f_2\| \leq \epsilon$, but $\|g_1^* - g_2^*\| = 1$.

From the compact, Hausdorff, and infinite nature of X, it follows that X has a condensation point x^*. Select two independent elements g^1 and g^2 of G and let

$$\bar{g} = g_2(x^*) \cdot g_1 - g_1(x^*) \cdot g_2,$$

thus $\bar{g}(x^*) = 0$. We may assume $\|\bar{g}\| = 1$. By continuity there is a neighborhood N of x^* on which $|\bar{g}| < \epsilon$. Since N is a neighborhood of a condensation point, we may select $n+1$ distinct points $\{x_j\}_{j=1}^{n+1}$ in N. Furthermore, since X is compact and Hausdorff, hence normal, there exists an open subset N_1 of N such that $\overline{N}_1 \subset N$ and the points $\{x_j\} \subset N_1$.

Let $\{r_j\}_{j=1}^{n+1}$ be a set of $n+1$ real values such that $r_1 = 1$ and

$$\sum_{j=1}^{n+1} r_j g(x_j) = 0$$

for every $g \in G$ (that such is possible follows immediately from the Chebyshev character of G). Furthermore, from Cheney [p. 41], it seen that if

$$f_1(x_j) = \operatorname{sgn} r_j, \qquad j = 1,..., n+1,$$

then the best approximation to f_1 on $\{x_j\}_{j=1}^{n+1}$ is the zero function. To see this let $\tilde{g}$ be the best least squares approximation to f_1 and $\tilde{r} = f_1 - \tilde{g}$, then

$$\sum_{j=0}^{n} \tilde{r}(x_j)\, g(x_j) = 0 \qquad \text{for all} \quad g \in G,$$

and (since $\{\tilde{r}(x_j)\}$ and $\{r_j\}$ are both nonzero $n+1$-vectors orthogonal to n dimensional G) for some $\alpha \neq 0$

$$r_j = \alpha \tilde{r}(x_j) \text{ for all } j.$$

Hence

$$\begin{aligned} r_j^2 = \alpha^2 \tilde{r}(x_j)^2 &= \alpha r_j(f_1(x_j) - \tilde{g}(x_j)) \\ &= \alpha r_j(\operatorname{sgn} r_j - \tilde{g}(x_j)) \\ &= \alpha \,|\, r_j \,|\, - \alpha r_j\, \tilde{g}(x_j) \\ &= \alpha \,|\, \alpha \,|\, |\, \tilde{r}(x_j)| - \alpha^2 \tilde{r}(x_j)\, \tilde{g}(x_j). \end{aligned}$$

Thus

$$\begin{aligned} \alpha^2 \sum_{j=0}^{n} \tilde{r}(x_j)^2 &= \alpha \,|\, \alpha \,| \cdot \sum_{j=0}^{n} |\, \tilde{r}(x_j)| - \alpha^2 \sum_{j=0}^{n} \tilde{r}(x_j)\, \tilde{g}(x_j) \\ &= \operatorname{sgn} \alpha \cdot \alpha^2 \cdot \sum_{j=0}^{n} |\, \tilde{r}(x_j)|, \end{aligned}$$

which implies

$$\epsilon = \frac{\sum_{j=0}^{n} (\tilde{r}(x_j))^2}{\sum_{j=0}^{n} |\, \tilde{r}(x_j)|} = 1.$$

We have then

$$\operatorname{sgn}(f_1(x_j) - 0) = \operatorname{sgn} r_j = \operatorname{sgn} \tilde{r}(x_j) \cdot \epsilon.$$

Using the Tietze extension theorem, extend f_1 to X such that $\| f_1 \| = 1$ and $f_1 = 0$ on $X \sim N_1$. Thus $\| f_1 \| = \| 0 - f_1 \| = 1$, so 0 is also the best approximation to f_1 on X.

Now define $f_2 = \tilde{g}$ on the boundary of N_1 (i.e., on $\overline{N}_1 \cap X \sim N_1$), $f_2 = 0$ on $X \sim N$, and extend f_2 to the compact set $X \sim N_1$ such that $| f_2 | \leqslant \epsilon$

(recall $|\bar{g}| < \epsilon$ on N_1, hence $\bar{N}_1$). On $\bar{N}_1 \cap X \sim N_1$ the function $f_2 = \bar{g} = f_1 + \bar{g}$, hence we may extend f_2 in a continuous fashion to all of X by letting $f_2 = f_1 + \bar{g}$ on N_1.

Since $|f_2 - \bar{g}| = |f_1| \leq 1$ on N_1, $|f_2 - \bar{g}| = 2\epsilon < 1$ (assume $\epsilon < \frac{1}{2}$) on $N \sim N_1$, and $|f_2 - \bar{g}| = |\bar{g}| \leq 1$ on $X \sim N$: $\|f_2 - \bar{g}\| = 1$. Furthermore

$$f_2(x_j) - \bar{g}(x_j) = f_1(x_j) = \operatorname{sgn} r_j, \quad j = 1, \dots, n+1.$$

Thus $\bar{g}$ is the best approximation to f_2 on $\{x_j\}$, and since $\|f_2 - \bar{g}\| = 1$, $\bar{g}$ is also the best approximation to f_2 on X.

We have then that $g_1^* = 0$ and $g_2^* = \bar{g}$, thus $\|g_1^* - g_2^*\| = 1$. The proof will be complete if it is shown that $\|f_2 - f_1\| \leq \epsilon$. But on N_1, $|f_1 - f_2| = |\bar{g}| < \epsilon$, and on $X \sim N_1$, $|f_1 - f_2| \leq |f_2| \leq \epsilon$, thus $\|f_2 - f_1\|$ is bounded by ϵ. ∎

It should be noticed that the assumption that $n \geq 2$ was necessary to produce a $\bar{g}$ with a zero, but not identically zero. No such functions exist for one-dimensional Chebyshev systems.

3. The Construction of a Suitable K

In this section we seek to exhibit a technique for determining a constant K in inequality (1) and a K^* in (3). The construction of an appropriate K will involve the solution of $n + 1$ interpolation problems.

In Lemma 1 it was remarked that the quantity

$$K = [\min_{g \in G_1} \max_{x \in E} (\operatorname{sgn}[f(x) - g^*(x)]) \cdot g(x)]^{-1}$$

would suffice in (1). In Cheney's proof it is clear that the maximum could be taken on any subset $E_0 = \{x_i\}_{i=1}^{k}$ of E with the property that there exist positive scalars $\{\theta_i\}_{i=1}^{k}$ such that

$$0 = \sum_{j=1}^{k} \theta_j \operatorname{sgn}(f(x_j) - g^*(x_j)) \cdot g(x_j)$$

for every $g \in G$. Any $n + 1$ point subset of E on which g^* is the best approximation to f is such a set E_0. We assume a suitable E_0 has been selected and let

$$\bar{K} = [\min_{g \in G_1} \max_{x \in E_0} (\operatorname{sgn}[f(x) - g^*(x)]) \cdot g(x)]^{-1}. \tag{4}$$

In Lemma 3 it will be shown that $\overline{K}$ may be determined by taking minima on a far smaller set than G_1. To this end, for $g \in G$ let

$$\gamma(g) = \max_{x \in E_0} [\operatorname{sgn}(f(x) - g^*(x))] \cdot g(x),$$

and

$$G_1^* = \{ g \in G_1 : \gamma(g) = \operatorname{sgn}(f(x) - g^*(x)) \cdot g(x) \text{ for at least } n \text{ values } x \text{ of } E_0 \}.$$

First, it will be shown that there are $n + 1$ elements of G_1^*.

LEMMA 2. *G_1^* contains $n + 1$ members.*

Proof. Let $E_0 = \{x_j\}_{j=1}^{n+1}$. From Cheney [pp. 36, 41], we know there exist $n + 1$ positive scalars $\{\lambda_j\}_{j=1}^{n+1}$ such that

$$0 = \sum_{j=1}^{n+1} \lambda_j (\operatorname{sgn}[f(x_j) - g^*(x_j)]) \cdot g(x_j) \tag{5}$$

for all $g \in G$. Define $g_i \in G$ such that for $i = 1, \ldots, n + 1$

$$g_i(x_j) = \operatorname{sgn}[f(x_j) - g^*(x_j)], \qquad j = 1, \ldots, n + 1; \quad j \neq i.$$

Then

$$\operatorname{sgn}[f(x_i) - g^*(x_i)] \cdot g_i(x_i) = -1/\lambda_i \sum_{j \neq i} \lambda_j < 0$$
$$< 1 = \operatorname{sgn}[f(x_j) - g^*(x_j)] \cdot g_i(x_j)$$

for $j \neq i$. Thus $g_i' = g_i / \| g_i \| \in G_1^*$, and G_1^* has at least $n + 1$ elements. Now suppose g is any element of G_1^*; we seek to show $g = g_i'$ for some i, which would imply G_1^* has exactly $n + 1$ members. From Eq. (5), $\operatorname{sgn} g(x_i) = -\operatorname{sgn}[f(x_i) - g^*(x_i)]$ for some i and hence for $j \neq i$

$$\gamma(g) = \operatorname{sgn}(f(x_j) - g^*(x_j)) \cdot g(x_j);$$

thus g is a positive scalar multiple of g_i', but this multiple is 1 and $g = g_i'$ since $\| g \| = 1 = \| g_i' \|$. ∎

LEMMA 3. *The quantity $\overline{K}$ in* (4) *satisfies*

$$\overline{K} = [\min_{g \in G_1^*} \gamma(g)]^{-1}.$$

Proof. It will be shown that if $g \in G_1 \sim G_1^*$, then there is a $g' \in G_1$ such that $\gamma(g') < \gamma(g)$. Let $J = \{x \in E_0 : \gamma(g) = \operatorname{sgn}[f(x) - g^*(x)] \cdot g(x)\}$. Let us assume initially the existence of an $x^* \notin J$ such that $|g(x^*)| = \|g\| = 1$ (the case where $|g(x)| < 1$ for $x \notin J$, i.e., $\{x : |g(x)| = 1\} \subset J$, will be considered later).

Since $g \notin G_1^*$, J contains at most $n - 1$ points and we may determine an element $h \in G$ such that

$$h(x) = 0, \qquad \text{for} \quad x \in J$$

and

$$h(x^*) = g(x^*).$$

Then let $g_1 = g + \lambda h$ (λ to be specified later). For $x \in J$

$$\operatorname{sgn}[f(x) - g^*(x)]\, g_1(x) = \operatorname{sgn}[f(x) - g^*(x)]\, g(x) = \gamma(g);$$

for $x \in E_0 \sim J$

$$\begin{aligned}\operatorname{sgn}[f(x) - g^*(x)]\, g_1(x) &= \operatorname{sgn}[f(x) - g^*(x)]\, g(x) + \lambda \operatorname{sgn}[f(x) - g^*(x)]\, h(x)\\ &\leq \operatorname{sgn}[f(x) - g^*(x)] \cdot g(x) + |\lambda|\, \|h\| < \gamma(g)\end{aligned}$$

for λ sufficiently small. Thus $\gamma(g_1) = \gamma(g)$, yet

$$\|g_1\| \geq |g_1(x^*)| = |g(x^*) + \lambda g(x^*)| = 1 + \lambda > 1$$

for λ positive. Letting $g' = g_1/\|g_1\|$, it is seen that $g' \in G_1$ and

$$\gamma(g') = 1/\|g_1\|\, \gamma(g_1) < \gamma(g_1) = \gamma(g).$$

We have delayed consideration of the case where $\{x : |g(x)| = 1\} \subset J$, but for such a case the quantity $\gamma(g)$ must be unity. But 1 is also the upper bound for $\gamma(g)$ for $g \in G_1$, hence unless $\min_{g \in G_1} \gamma(g) = \max_{g \in G_1} \gamma(g)$, there will be a $g' \in G_1$ such that $\gamma(g') < \gamma(g)$ If maximum and minimum are equal, the nonemptiness of G_1^* suffices to show the lemma holds. ∎

Using Lemmas 2 and 3, the following is obvious.

THEOREM 5. *Let E_0 be an $n + 1$-point subset of E on which g^* is the best approximation to f. Writing $E_0 = \{x_j\}_{j=1}^{n+1}$, let, for $i = 1,\ldots, n + 1$, $g_i \in G$ be a function such that*

$$g_i(x_j) = \operatorname{sgn}[f(x_j) - g^*(x_j)], \qquad j = 1,\ldots, n + 1; \quad j \neq i.$$

Then the constant in inequality (1) *can be taken as*

$$\bar{K} = \max_i \|g_i\|.$$

Proof. In the proof of Lemma 2 it was shown that

$$\gamma(g_i) = 1$$

and that

$$g_i' = g_i/\|g_i\| \in G_1^*.$$

Thus

$$\gamma(g_i') = 1/\|g_i\|$$

and from Lemma 3

$$\bar{K} = [\min_i \gamma(g_i')]^{-1} = \max_i \|g_i\|. \quad ∎$$

Now we turn to the determination of the Lipschitz constant K^* in (3) assuming X is a finite set. As was shown in the proof of Theorem 3, it will suffice to consider each f in the unit ball of $C(X)$ and compute the corresponding $\bar{K}$, then let K^* be the maximum of all such $\bar{K}$. As can be seen from Theorem 5, $\bar{K}$ may be determined with the information of E_0 and $\text{sgn}[f - g^*]$ on E_0.

For a given $n + 1$-point subset E_0 of X, the values of $\text{sgn}[f - g^*]$ on E_0 can be determined as in the proof of Theorem 4. That is, let $E_0 = \{x_j\}_{j=1}^{n+1}$, and determine an $n + 1$-vector $\{r_j\}_{j=1}^{n+1}$ such that

$$0 = \sum_{j=1}^{n+1} r_j g(x_j) \qquad \text{and} \qquad r_1 = 1.$$

By considering a basis for G this involves solving an $n \times n$ algebraic system. It follows (see Cheney [p. 41]) that the best approximation to an f on E_0 will have residuals with sign $\text{sgn}[f(x_j) - g^*(x_j)] = +\text{sgn}\, r_j\, j = 1,..., n + 1$ or $\text{sgn}[f(x_j) - g^*(x_j)] = -\text{sgn}\, r_j\, j = 1,..., n + 1$. It then can be seen that $\bar{K}$ may be determined by solving the $n + 1$ interpolation problems as in Theorem 5, with right-hand sides taken from $\{\text{sgn}\, r_j\}_{j=1}^{n+1}$. (The cases for right-hand sides $\{-\text{sgn}\, r_j\}_{j=1}^{n+1}$ is handled concurrently since the quantities $\|g_i\|$ are independent of $\pm g_i$.) Since each $f \in C(X)$ has an E_0, by considering all $\binom{m}{n+1}$ such sets E_0 (m is the cardinality of X), computing the corresponding $\bar{K}$'s, and letting K^* be their maximum, the Lipschitz constant of (3) can be determined.

4. A Specific Example

We use the theory of Sec. 3 to determine a Lipschitz constant K^* where for fixed n, X is the set of $n + 1$ Chebyshev points $x_j = \cos j\pi/n$, $j = 0,..., n$, on the interval $[-1, 1]$ and G is the set of polynomials of degree $\leqslant n - 1$. It is easy to see that this same K^* will satisfy inequality (1) for $f = x^n$. Since X contains exactly $n + 1$ points, it coincides with the one possible set E_0. The signs of the residual are clearly $(-1)^j$, $j = 0,\dots, n$ (or $-(-1)^j$, $j = 0,\dots, n$). We need determine for $i = 0,\dots, n$, polynomials $g_i \in G$ such that

$$g_i(x_j) = (-1)^j, \qquad j = 0,\dots, n, j \neq i. \tag{6}$$

Let g_i be expanded in Chebyshev polynomials, i.e.,

$$g_i = \sum_{k=0}^{n-1} \alpha_k^i \cos(k \cos^{-1} x),$$

and using the transformation $\cos\theta = x$, we may seek cosine polynomials h_i such that

$$g_i(\cos\theta) = h_i(\theta) = \sum_{k=0}^{n-1} \alpha_k^i \cos k\theta.$$

The conditions (6) now become

$$h_i(j\pi/n) = (-1)^j, \qquad j = 0,\dots, n, j \neq i.$$

Consider

$$h_i = (-1)^{i+1} \left(1 + 2 \sum_{k=1}^{n-1} \cos k \frac{i\pi}{n} \cos k\theta\right),$$

using simple trigonometric identities, it can be shown that

$$h_i = \cos n\theta - \frac{\sin n\theta \sin\theta}{\cos\theta - \cos i\pi/n}$$

from which it is clear that

$$h_i(j\pi/n) = (-1)^j, \qquad j \neq i,$$

and thus the solution to the interpolation problem is

$$g_i = (-1)^{i+1} \left(1 + 2 \sum_{k=1}^{n-1} \cos k \frac{i\pi}{n} \cos(k \cos^{-1} x)\right).$$

From this representation it follows immediately that $\|g_i\| \leq 2n - 1$, and in fact for $i = n$

$$g_n(-1) = (-1)^{n+1} \left(1 + 2 \sum_{k=1}^{n-1} (-1)^k (-1)^k\right).$$

Thus

$$\|g_n\| = 2n - 1.$$

We have succeeded in showing then that

$$K^* = \max \|g_i\| = 2n - 1.$$

5. The Complex Case

Along with inequality (1) for real approximation problems, Newman and Shapiro present a complex version:

$$\|g - g^*\| \leq K_1(\|f - g\| - \|f - g^*\|)^{1/2} + K_2(\|f - g\| - \|f - g^*\|). \quad (7)$$

Since this inequality is poorer than one of the form (1), it becomes of interest to determine if it can be improved. To be precise, does estimate (1) hold for the complex approximation case?

The following simple example shows that such is not the case. An f and a sequence $\{g_j\}$ are exhibited such that for no finite K does

$$\|g_j - g^*\| \leq K(\|f - g_j\| - \|f - g^*\|)$$

hold for all j.

Let $X = \{-1\} \cup \{+1\}, f(1) = 1, f(-1) = -1$, and G be the one-dimensional space of complex constants. Then $g^* = 0$ and $\|f - g^*\| = 1$. But for $g_j = i \cdot (1/j)\ j = 1, 2,...$

$$\|g_j - g^*\| = 1/j \qquad \text{and} \qquad \|f - g_j\| = \frac{\sqrt{1 + j^2}}{j}.$$

Thus

$$\frac{\|f - g_j\| - \|f - g^*\|}{\|g^* - g_j\|} = \sqrt{1 + j^2} - j,$$

which cannot be uniformly bounded from below by a positive constant.

Although estimate (1) may not hold in the complex case, it is possible to use (7) to produce a continuity condition for the complex case similar to (2).

Theorem 6. *For the complex approximation problem (with notation as before) there exist constants K_1^f and K_2^f depending upon f_1 such that*

$$\|g_1^* - g_2^*\| \leq K_1^f \|f_1 - f_2\|^{1/2} + K_2^f \|f_1 - f_2\|.$$

Proof. From (7) we have

$$\|g_1^* - g_2^*\| \le K_1(\|f_1 - g_2^*\| - \|f_1 - g_1^*\|)^{1/2} + K_2(\|f_1 - g_2^*\| - \|f_1 - g_1^*\|).$$

But

$$\begin{aligned}\|f_1 - g_2^*\| - \|f_1 - g_1^*\| &\le \|f_1 - f_2\| + \|f_2 - g_2^*\| - \|f_1 - g_1^*\| \\ &\le \|f_1 - f_2\| + \|f_2 - g_1^*\| - \|f_1 - g_1^*\| \\ &\le \|f_1 - f_2\| + \|f_2 - f_1\| + \|f_1 - g_1^*\| - \|f_1 - g_1^*\| \\ &= 2\|f_1 - f_2\|.\end{aligned}$$

Thus the theorem follows with

$$K_1' = \sqrt{2}\, K_1 \quad \text{and} \quad K_2' = 2K_2. \quad \blacksquare$$

References

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